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MATHEMATICAL MODELING OF EXTENSION OF AN INHOMOGENEOUS ELASTIC CLOTH

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We propose mathematical models for a knitted fabric in tension that can be used to calculate the strain of the cloth thread from the properties of the cloth and the loads applied to the cloth. Under simplifying assumptions, the problem is reduced to consideration of an elementary cell of the cloth that contains a thread loop. The loop is first modeled by a thread oval with point forces and then by a plate with a hole under distributed loads. In strength analysis, this allows one to use methods of elasticity theory. A number of hypotheses are used to establish the relation between the stress state of the thread oval and the corresponding state of the plate, which makes it possible to model the mechanical behavior of a thread in the material in a variety of forms. The theoretical relationships obtained are compared to the experimental data available in the literature.

In the manufacture of textile materials from new types of raw materials and in the design of knitted articles meeting up-to-date requirements, there is a need to solve a wide class of problems, among which strength analysis plays an important role. Knowledge of the deformation-strength properties of starting materials allows one to choose an optimum material and to predict the behavior of articles during operation.

Knitted fabrics form an important class of textile articles. They can be regarded as a discrete medium consisting of cells connected by loops or knots. These fabrics have been widely used in various applications. Development and improvement of methods for the strength analysis of these materials have been the subject of many theoretical and experimental studies [1–5].

In the present paper, using methods of continuum mechanics, we develop a mathematical model for a knitted fabric that establishes the relation among the strain of a thread in the fabric, the parameters of the fabric, and the load applied to it. This relation can be used, in particular, to obtain tensile diagrams, which are used in designing textile articles.

1. A knitted fabric has a complicated structure. Its mechanical properties are determined by various qualitative and quantitative factors. In the fabric structure, elementary cells having the shape of loops can be arranged in one or several layers and, generally, can have different shapes and orientations. Moreover, the elementary cells can themselves change with changes in the length of the loop, thickness of the thread, etc. Neglecting secondary factors, we adopt the following assumptions:

— a knitted fabric is a regular system of cells (loops) arranged in one layer (Fig. 1);

— all loops have the same dimensions and shape, and, within an elementary link, the thread has the same thickness and deformation-strength properties;

— the lateral surfaces of the fabric are subjected to in-plane loads of constant intensity and the faces are free of tractions;

— the flexural rigidity of the thread, the friction at the spots of contact of elementary links, and the weight of the fabric are neglected;

— the deformation is quasistatic and proceeds at constant humidity and temperature.

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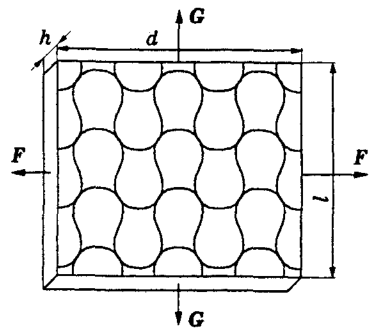

Fig. 1

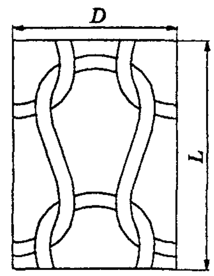

Fig. 2

Fig. 1. Structure of a knitted fabric.

Fig. 2. Structure of a typical cell.

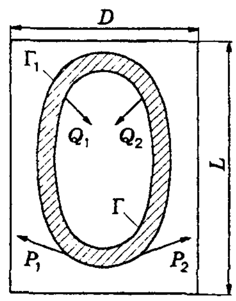

Fig. 3

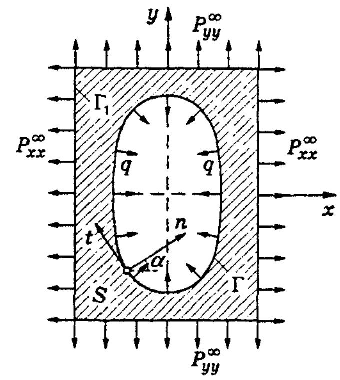

Fig. 4

Fig. 3. Modeling of a loop by an oval with point loads.

Fig. 4. Modeling of a loop by a plate with distributed loads.

We consider a rectangular sheet of a knitted fabric with width d, length l, and thickness h ($l \sim d$ and $h \ll d$). The cloth is stretched in its plane by a longitudinal force $\boldsymbol{G}$ and a transverse force $\boldsymbol{F}$ (Fig. 1). From the specified loads and the parameters of the cloth, we determine the tensile stress in the thread in a state of equilibrium and establish the relation between the stress and strain of the thread.

The sheet of the material is a system of elementary links, whose interaction is determined by the applied loads, the fabric structure, and the properties of the thread. We divide the sheet into w longitudinal and v transverse strips with dimensions D (loop width) and L (height of the loop row), respectively:

$$D = d/w, \qquad L = l/v. \tag{1}$$

We consider a typical elementary cell containing a thread loop (Fig. 2).

At the first stage of modeling, modifying the results of [1], we assume that the thread loop is an oval with outer Γ and inner Γ_1 contours loaded by lateral point forces: the normal reactions of the links of the previous loop row $\boldsymbol{Q}_\nu$ and the next loop row and the reactions of the discarded parts of the loop considered $\boldsymbol{P}_\nu$ (Fig. 3).

It should be noted that the shape of the oval, the forces acting on the oval, and the points at which these forces are applied are not known in advance. Moreover, in quasistatical extension of the sheet, the points of contact of neighboring links and the points of application of the forces are shifted, and this complicates analysis.

To overcome the above-mentioned difficulties and develop a method for determining tension in the thread, we describe the second stage of modeling, at which we replace the discrete distribution of mass in the cell (along the thread loop) and the discrete distribution of forces on the oval (at different points of the lateral surface) by continuous distributions.

We model the thread oval by a material layer, i.e., a plate of the same mass with a hole, whose inner contour coincides with the inner contour of the oval and whose outer contour coincides with the contour of the rectangle that bounds the cell (Fig. 4). The discrete load is modeled by an equivalent distributed contour load of constant intensity. The forces that are internal with respect to the oval $\boldsymbol{Q}_\nu$ are modeled by distributed normal load q acting on the hole contour in the plate. The external forces $\boldsymbol{P}_\nu$ are replaced by equivalent distributed normal loads P_{xx}^∞ and P_{yy}^∞ acting on the longitudinal and transverse sides of the rectangle, respectively, where x and y are Cartesian axes (Fig. 4).

The unknown contour load q is found after the stresses in the plate are determined. The peripheral loads P_{xx}^∞ and P_{yy}^∞ can be found from the loads applied at the sheet and from the geometric parameters of the sheet. Indeed, the forces on the sides of the rectangular cell are given by the relations

$$P_{xx}^\infty Lh = F/v, \qquad P_{yy}^\infty Dh = G/w,$$

whence, with allowance for (1), the desired loads are

$$P_{xx}^\infty = F/(hl), \qquad P_{yy}^\infty = G/(hd). \tag{2}$$

In what follows, we assume that the stress state of the thread oval is determined by stresses in the plate and is expressed in terms of the stresses using a certain hypothesis. Thus, strength analysis for the thread loop reduces to analysis for a plate with a hole (treated as an infinite plane with a hole) subjected to normal contour and peripheral loads, for which methods of elasticity theory can be used.

2. In accordance with the initial assumptions, for a typical element of the sheet, body forces are absent, the faces are free of stresses, and the load on the lateral surface is parallel to its plane. Consequently, both the element and the modeling plate are under the generalized plane stresses described by the two-dimensional elastic problem [6].

Assuming that the applied loads cause small strains of the thread and the stress state of the thread loop is uniform, we calculate the stresses in the plate with a hole using the linear theory of elasticity and assuming a full-strength hole contour.

In this theory, the two-dimensional static problem reduces to equations of equilibrium, Hooke's law, and strain–displacement relations. In the absence of body forces, these relations in the Cartesian coordinates $x_1 = x$ and $x_2 = y$ ($\partial_1 = \partial x_1$, $\partial_2 = \partial x_2$) have the form [6]

$$\begin{gathered} \partial_\beta P_{\alpha\beta} = 0, \qquad P_{\alpha\beta} = \lambda_0 \varepsilon_{\sigma\sigma}\delta_{\alpha\beta} + 2\mu\varepsilon_{\alpha\beta}, \qquad 2\varepsilon_{\alpha\beta} = \partial_\alpha u_\beta + \partial_\beta u_\alpha, \\ \lambda_0 = 2\lambda\mu/(\lambda + 2\mu) \qquad (\alpha, \beta, \sigma = 1,\ 2). \end{gathered} \tag{3}$$

Here $\delta_{\alpha\beta}$ is the Kronecker symbol, λ and μ are the Lamé coefficients of elasticity, and u_α, $\varepsilon_{\alpha\beta}$, and $P_{\alpha\beta}$ are components of displacements, strains, and stresses, respectively, which are averaged across the thickness of the plate and are functions of the x_1 and x_2 coordinates:

$$u_\alpha(x_1, x_2) = \frac{1}{h}\int\limits_{-h/2}^{h/2} u_\alpha(x_1, x_2, x_3)\, dx_3,$$

$$\varepsilon_{\alpha\beta}(x_1, x_2) = \frac{1}{h}\int\limits_{-h/2}^{h/2} \varepsilon_{\alpha\beta}(x_1, x_2, x_3)\, dx_3, \qquad P_{\alpha\beta}(x_1, x_2) = \frac{1}{h}\int\limits_{-h/2}^{h/2} P_{\alpha\beta}(x_1, x_2, x_3)\, dx_3.$$

Here summation is performed over repeated indices [the stresses P_{13} and P_{23} are small in comparison with

P_{11}, P_{12}, and P_{22}, $P_{33} = 0$, and the strain ε_{33} is expressed in terms of ε_{11} and ε_{22} by the formula $\varepsilon_{33} = -\lambda(\varepsilon_{11} + \varepsilon_{22})/(\lambda + 2\mu)$].

A distinguishing feature of a full-strength contour is that the stress concentration at each of its point is the same. This contour loses strength simultaneously at all points, and this occurs as a rule at large loads. In the case considered, the following boundary conditions correspond to the full-strength contour:

$$P_{11} = P_{xx}^{\infty}, \qquad P_{22} = P_{yy}^{\infty}, \qquad P_{12} = 0 \quad \text{on} \quad \infty; \tag{4}$$

$$p_n = P_{nn} = q = \text{const}, \quad p_t = P_{nt} = 0, \quad P_{tt} = \sigma = \text{const on } \Gamma, \quad q = -k\sigma \ (0 < k < 1). \tag{5}$$

Here p_n and p_t and P_{nn}, P_{nt}, and P_{tt} are components of the stress vector and tensor, respectively, in the natural axes of the contour Γ (normal n and tangent t) (Fig. 4). Here we assume that the stress q is part of the stress σ and has the opposite sign. The values of P_{xx}^{∞}, P_{yy}^{∞}, k λ, and μ in (3)–(5) are specified and the constants q and σ, the contour shape, and the stresses are to be determined.

Relations (3) lead to the equations of equilibrium and the compatibility equation in stresses $\partial_\beta P_{\alpha\beta} = 0$ and $(\partial_{11} + \partial_{22})(P_{11} + P_{22}) = 0$ $(\alpha, \beta = 1, 2)$. In the complex variables, $z^1 = z = x + iy$ and $z^2 = \bar{z} = x - iy$ $(\partial_z = \partial/\partial z$ and $\partial_{\bar{z}} = \partial/\partial\bar{z})$, these equations have the form

$$\partial_z P^{11} + \partial_{\bar{z}} P^{12} = 0, \qquad \partial_{z\bar{z}} P^{12} = 0. \tag{6}$$

Here $P^{\alpha\beta}$ are the complex components of stresses, which are related to the Cartesian components by the transformation formulas $P^{\alpha\beta} = P_{\sigma\tau}(\partial z^\alpha/\partial x_\sigma)(\partial z^\beta/\partial x_\tau)$ as follows:

$$P^{11} = \bar{P}^{22} = P_{xx} - P_{yy} + 2iP_{xy}, \qquad P^{12} = P^{21} = P_{xx} + P_{yy}.$$

The general solution of Eqs. (6) is expressed in terms of the complex potentials $\Phi(z)$ and $\Psi(z)$ by Kolosov's formulas [7]:

$$P^{11} = \bar{P}^{22} = -2[z\bar{\Phi}'(\bar{z}) + \bar{\Psi}(\bar{z})], \qquad P^{12} = P^{21} = 2[\Phi(z) + \bar{\Phi}(\bar{z})]. \tag{7}$$

In the infinite region S — the exterior of the hole in the plate — stresses should be single-valued and bounded. These requirements imply that the potentials have the form

$$\Phi(z) = A + \sum_{k=1}^{\infty} A_k z^{-k}, \qquad \Psi(z) = B + \sum_{k=1}^{\infty} B_k z^{-k}, \tag{8}$$

where the constants A and B can be considered real; they are expressed in terms of the peripheral loads (4):

$$A = (P_{yy}^{\infty} + P_{xx}^{\infty})/4, \qquad B = (P_{yy}^{\infty} - P_{xx}^{\infty})/2. \tag{9}$$

The contour conditions (5) imply boundary-value problems for the potentials and the equation of the hole contour. Regarding the natural axes n and t on the hole contour as the Cartesian axes x' and y' rotated through angle α about the x and y axes (Fig. 4) and introducing, in addition to z and $\bar{z}$, the complex coordinates $z' = x' + iy'$ and $\bar{z}' = x' - iy'$, we obtain

$$z'^1 = z' = z^1 e^{-i\alpha}, \qquad z'^2 = \bar{z}' = z^2 e^{i\alpha}.$$

Consequently, the complex stresses $P'^{\alpha\beta}$ and $P^{\sigma\tau}$ in the corresponding variables are related to one another by the transformation formulas $P'^{\alpha\beta} = P^{\sigma\tau}(\partial z'^\alpha/\partial z^\sigma)(\partial z'^\beta/\partial z^\tau)$:

$$P_{x'x'} + P_{y'y'} = P'^{12} = P^{12}, \qquad P_{x'x'} - P_{y'y'} + 2iP_{x'y'} = P'^{11} = P^{11} e^{-2i\alpha} \quad \text{on} \quad \Gamma.$$

The angle α can be found from the equation of the contour $z = z(s)$ and $\bar{z} = \bar{z}(s)$. Indeed, the unit vector tangent to the contour is given by the formula $dz/ds = e^{i(\pi/2+\alpha)}$, whence the desired angle is obtained in the form $e^{-2i\alpha} = -d\bar{z}/dz$ on Γ. Thus, the relation between the natural and complex components of stresses on the contour (with allowance for the coincidence of the axes $x' = n$ and $y' = t$) is given by the formulas

$$P_{nn} + P_{tt} = P^{12}, \quad P_{nn} - P_{tt} + 2iP_{nt} = -\frac{d\bar{z}}{dz} P^{11} \quad \text{on} \quad \Gamma.$$

Finally, substitution of stresses (7) and the boundary values (5) into the last equations leads to the following boundary-value problems for the potentials and the problem for the equation of the contour:

$$2[\Phi(z) + \bar{\Phi}(\bar{z})] = q + \sigma, \quad 2\,\frac{dz}{d\bar{z}}\,[\bar{z}\Phi'(z) + \Psi(z)] = q - \sigma \quad \text{on} \quad \Gamma. \tag{10}$$

To solve these problems, we map conformally the exterior of the hole S onto the exterior S' of a circle with unit radius on an auxiliary plane ζ (with correspondence of infinitely remote points) using the holomorphic function

$$z = w(\zeta) = c\zeta + w_0(\zeta), \quad c = \bar{c}, \quad w_0(\zeta) = \sum_{k=0}^{\infty} c_k \zeta^{-k}, \quad \zeta = re^{i\theta} \in S'. \tag{11}$$

The potentials (8) are then functions of the variable ζ:

$$\Phi(\zeta) = A + \sum_{k=1}^{\infty} a_k \zeta^{-k}, \qquad \Psi(\zeta) = B + \sum_{k=1}^{\infty} b_k \zeta^{-k}, \tag{12}$$

and the stresses (7) are calculated from the formulas

$$P^{11} = \bar{P}^{22} = -2\Big[w(\zeta)\,\frac{\bar{\Phi}'(\bar{\zeta})}{\bar{w}'(\bar{\zeta})} + \bar{\Psi}(\bar{\zeta})\Big], \qquad P^{12} = 2[\Phi(\zeta) + \bar{\Phi}(\bar{\zeta})]. \tag{13}$$

On the boundary Γ' of the unit circle $|\zeta| = 1$, we have

$$\zeta = e^{i\theta}, \qquad \bar{\zeta} = e^{-i\theta}, \qquad \bar{\zeta} = 1/\zeta.$$

$$dz = w'(\zeta)\,d\zeta = i\zeta w'(\zeta)\,d\theta, \quad d\bar{z} = -i\bar{\zeta}\bar{w}'(\bar{\zeta})\,d\theta, \quad \frac{dz}{d\bar{z}} = -\frac{\zeta}{\bar{\zeta}}\,\frac{w'(\zeta)}{\bar{w}'(\bar{\zeta})} = -\zeta^2\frac{w'(\zeta)}{\bar{w}'(\bar{\zeta})},$$

Therefore, the boundary-value problems (10) is written as

$$2[\Phi(\zeta) + \bar{\Phi}(\bar{\zeta})] = q + \sigma \quad \text{on} \quad \Gamma'; \tag{14}$$

$$2\zeta^2[\bar{w}(\bar{\zeta})\Phi'(\zeta) + w'(\zeta)\Psi(\zeta)] = (\sigma - q)\bar{w}'(\bar{\zeta}) \quad \text{on} \quad \Gamma'. \tag{15}$$

From (12) and (14) it follows that the analytic function $\Phi(\zeta)$ is bounded at infinity and its real part is constant on the unit circle. These conditions are satisfied by setting the function constant everywhere in the region S' (and equal to its value at infinity):

$$\Phi(\zeta) = A = (P_{xx}^{\infty} + P_{yy}^{\infty})/4. \tag{16}$$

Then, condition (14) defines the desired stress σ [and, with allowance for (5), the load q] in terms of the specified quantities:

$$\sigma = 4A/(1-k), \qquad q = -4Ak/(1-k). \tag{17}$$

From (17) it follows that $\sigma > 0$ and $q < 0$, i.e., the contour of the hole is stretched and subjected to an internal pressure.

By virtue of (16) and (17), condition (15) is simplified:

$$\zeta^2 w'(\zeta)\Psi(\zeta) = (2A - q)\bar{w}'(\bar{\zeta}), \qquad |\zeta| = 1.$$

Using analytic continuation, we write this condition in the form of a functional equation for the exterior of the circle:

$$\zeta^2 w'(\zeta)\Psi(\zeta) = H\bar{w}'(1/\zeta), \qquad \zeta \in S', \qquad H = 2A - q = 2A\,\frac{1+k}{1-k} \tag{18}$$

and employ it to determine the functions $w(\zeta)$ and $\Psi(\zeta)$.

We seek the function $w_0(\zeta)$ in the mapping (11) in the form of a polynomial of order $2l+1$ in odd powers of the argument with real coefficients: $w_0(\zeta) = \sum_{k=0}^{l} C_{2k+1}\zeta^{-(2k+1)}$ and $\bar{C}_{2k+1} = C_{2k+1}$; the mapping and its derivatives have the form

$$w(\zeta) = C\zeta + \sum_{k=0}^{l} C_{2k+1}\zeta^{-(2k+1)}, \qquad w'(\zeta) = C - \sum_{k=0}^{l}(2k+1)C_{2k+1}\zeta^{-(2k+2)},$$
$$\bar{w}'(1/\zeta) = C - \sum_{k=0}^{l}(2k+1)\zeta^{2k+2}. \tag{19}$$

Using (12) and (19), we infer that the left and right sides of Eq. (18) have the following orders at infinity:

$$\zeta^2 w'(\zeta)\Psi(\zeta) = O(\zeta^2), \qquad H\bar{w}'(1/\zeta) = O(\zeta^{2l+2}),$$

which coincide for $l = 0$. Consequently, the mapping (19) is determined by the following two-term expression containing two real parameters:

$$z = w(\zeta) = n(\zeta + m/\zeta), \qquad n = C, \qquad m = C_1/C. \tag{20}$$

The mapping allows us to determine the shape of the hole in the plate and, consequently, the shape of the oval. Indeed, according to the chosen mapping, the full-strength contour is the image of a unit circle. Setting $\zeta = e^{i\theta}$ for points on the circumference in (20), we obtain the following equation of full-strength contour:

$$x = a\cos\theta, \qquad y = b\sin\theta, \qquad a = n(1+m), \qquad b = n(1-m),$$
$$0 < n = (a+b)/2 < \infty, \qquad -1 < m = (a-b)/(a+b) < 1. \tag{21}$$

According to (21), the full-strength contour is an ellipse with semiaxes a and b. The center of the ellipse is at the coordinate origin, and the symmetry axes coincide with the Cartesian axes; the parameter m characterizes the shape of the ellipse, and the parameter n characterizes its dimensions.

By virtue of (20), the functional equation (18) defines the second potential:

$$\Psi(\zeta) = H\,\frac{1 - m\zeta^2}{\zeta^2 - m}. \tag{22}$$

For the potentials (16) and (22) and the mapping (20), the following relations for the stresses (13) hold:

$$P^{11} = \bar{P}^{22} = 2H\,\frac{m\bar{\zeta}^2 - 1}{\bar{\zeta}^2 - m}, \qquad P^{12} = 4A. \tag{23}$$

From (23) in the limit $\zeta \to \infty$, we obtain the equality $P^{11}_{\infty} = 2mH$, which, with allowance for (4), (9), and (18), defines the parameter m as a function of the specified quantities

$$m = -B/H = \chi(1-k)/(1+k), \qquad \chi = (P^{\infty}_{xx} - P^{\infty}_{yy})/(P^{\infty}_{xx} + P^{\infty}_{yy}). \tag{24}$$

Thus, of the two parameters m and n in the mapping (20), the first parameter is determined by the applied load and the second parameter remains constant, which implies the existence of a one-parameter family of similar ellipses. (This result was obtained in [8, 9] by different methods.) An ellipse from this family can be determined by specifying, for instance, one of its semiaxes. It follows from (24) that, for fixed peripheral loads, this formula gives an unambiguous relation between m and k, i.e., between the shape of the hole and the coefficient of proportionality of the contour stresses.

By virtue of (20), the polar coordinates r and θ in the plane of the unit circle correspond to the elliptic coordinates in the plane of the plate: the circumferences r = const correspond to ellipses and the radials θ = const to hyperbolas. The physical components of stresses in these coordinates P_{rr}, $P_{r\theta}$ and $P_{\theta\theta}$ are expressed in terms of the complex components and the mapping by the formulas [10]

$$P_{rr} - P_{\theta\theta} + 2iP_{r\theta} = \frac{\bar{\zeta}}{\zeta}\,\frac{\bar{w}'(\bar{\zeta})}{w'(\zeta)}\,P^{11}, \qquad P_{rr} + P_{\theta\theta} = P^{12}.$$

Using the mapping (20) and the stress relations (23), we find from the last formulas that the components and linear invariant of stresses in the plate in elliptic coordinates have the form

$$P_{rr} = 2A + H\,\frac{m(r^4+1)\cos 2\theta - (1+m^2)r^2}{r^4+m^2-2mr^2\cos 2\theta}, \qquad I = P_{rr} + P_{\theta\theta} = 4A, \tag{25}$$

$$P_{\theta\theta} = 2A - H\,\frac{m(r^4+1)\cos 2\theta - (1+m^2)r^2}{r^4+m^2-2mr^2\cos 2\theta}, \qquad P_{r\theta} = -H\,\frac{m(r^4-1)\sin 2\theta}{r^4+m^2-2mr^2\cos 2\theta}.$$

Thus, the stresses at a typical point of the plate depend on its coordinates, the applied load, and the shape of the hole; the stress invariant is constant everywhere in the plate. To determine the relation between the postulated uniform stresses in the oval and the variable stresses in the plate, we average the stresses over both coordinates.

We average the stresses over the variable $R = r^2$, which corresponds to the width of the transformed oval, in the infinite interval $1 \leqslant R \leqslant \infty$ using the formula

$$P^*(\theta) = \lim_{R\to\infty}\Big(\frac{1}{R-1}\int_1^R P(R,\theta)\,dR\Big).$$

To this end, we write equalities (25) in the form

$$P_{rr} = 2A - B\cos 2\theta + k_1, \quad P_{\theta\theta} = 2A + B\cos 2\theta + k_2, \quad P_{r\theta} = B\sin 2\theta + k_3, \quad I = 4A, \tag{26}$$

where $k_\sigma = H(\alpha_\sigma R + \beta_\sigma)/(R^2+bR+c)$ ($\sigma = 1$, 2, and 3). Here $\alpha_1 = -(1-m^2\cos 4\theta)$, $\beta_1 = m(1-m^2)\cos 2\theta$, $R = r^2$, $\alpha_2 = 1 - m^2\cos 4\theta$, $\beta_2 = -m(1-m^2)\cos 2\theta$, $b = -2m\cos 2\theta$, $\alpha_3 = -m^2\sin 4\theta$, $\beta_3 = m(1+m^2)\sin 2\theta$, and $c = m^2$. We note that the discriminant γ of the quadratic trinomial in the denominator in the expression for k_σ is negative:

$$\gamma = b^2 - 4c = -4m^2\sin^2 2\theta < 0.$$

The value of the function R averaged over k_σ in the finite interval $(1, R)$ calculated in accordance with [11] for $\gamma < 0$ is given by the expression

$$\frac{1}{R-1}\int_1^R k_\sigma(R,\theta)\,dR = \frac{H}{R-1}\int_1^R \frac{\alpha_\sigma R + \beta_\sigma}{R^2+bR+c}\,dR$$

$$= \frac{H}{R-1}\Big[\frac{\alpha_\sigma}{2}\ln\frac{R^2+bR+c}{1+b+c} + \frac{2\beta_\sigma - b\alpha_\sigma}{\sqrt{-\gamma}}\Big(\arctan\frac{2R+b}{\sqrt{-\gamma}} - \arctan\frac{2+b}{\sqrt{-\gamma}}\Big)\Big].$$

Passing to the limit $R \to \infty$ and evaluating the indeterminate form, we infer that the average value of the function in the infinite interval vanishes:

$$k_\sigma^*(\theta) = \lim_{R\to\infty}\Big[\frac{1}{R-1}\int_1^R k_\sigma(R,\theta)\,dR\Big] = H\lim_{R\to\infty}\Big[\frac{\alpha_\sigma}{2}\,\frac{2R+b}{R^2+bR+c} + 2\,\frac{2\beta_\sigma - b\alpha_\sigma}{(2R+b)^2-\gamma}\Big] = 0. \tag{27}$$

In view of (27), the stresses (26) (which can be assumed to be determined on the hole contour) averaged over R have the form

$$P^*_{rr} = 2A - B\cos 2\theta, \quad P^*_{\theta\theta} = 2A + B\cos 2\theta, \quad P^*_{r\theta} = B\sin 2\theta, \quad I^* = 4A. \tag{28}$$

The stresses are functions of θ and do not depend on the shape of the hole.

Subsequent averaging of the stresses (28) over the variable θ, i.e., along the length of the oval, using the formula

$$P^{**} = \frac{1}{2\pi}\int_0^{2\pi} P^*(\theta)\,d\theta$$

leads to the uniform field of average stresses with zero shear stress:

$$P_{rr}^{**} = 2A, \quad P_{\theta\theta}^{**} = 2A, \quad P_{r\theta}^{**} = 0, \quad I^{**} = 4A. \tag{29}$$

We note that on the contour of the hole ($r = 1$) the nonaveraged stresses in the elliptic coordinates and their invariant (25) are constant together with the average stresses:

$$\begin{gathered} P_{rr}^1 = 2A - H = q = -k\sigma, \qquad P_{\theta\theta}^1 = 2A + H = \sigma, \qquad P_{r\theta}^1 = 0, \\ I^1 = P_{rr}^1 + P_{\theta\theta}^1 = 4A = (1-k)\sigma \end{gathered} \tag{30}$$

[σ is given by formula (17)]. Here and below, the values corresponding to $r = 1$ are denoted by the superscript 1.

Thus, the stress field of the plate can be characterized by the stress invariant (25), or the average stresses (29), or the contour stresses (30). Using Hooke's law for a plate, we find the relation between the above-mentioned stresses and the corresponding strains.

We write Hooke's law in the elliptic coordinates as

$$P_{rr} = \lambda_0 J + 2\mu\varepsilon_{rr}, \quad P_{\theta\theta} = \lambda_0 J + 2\mu\varepsilon_{\theta\theta}, \quad P_{r\theta} = 2\mu\varepsilon_{r\theta}, \tag{31}$$

whence

$$I = 2(\lambda_0 + \mu)J, \qquad I = P_{rr} + P_{\theta\theta}, \qquad J = \varepsilon_{rr} + \varepsilon_{\theta\theta}. \tag{32}$$

The average stresses (29) correspond to the strains and ε_{rr}^{**}, $\varepsilon_{\theta\theta}^{**}$, and $\varepsilon_{r\theta}^{**}$ and their invariant J^{**}. Averaging the law (31) over both coordinates leads to the following relations between the average quantities:

$$P_{rr}^{**} = \lambda_0 J^{**} + 2\mu\varepsilon_{rr}^{**}, \quad P_{\theta\theta}^{**} = \lambda_0 J^{**} + 2\mu\varepsilon_{\theta\theta}^{**}, \quad P_{r\theta}^{**} = 2\mu\varepsilon_{r\theta}^{**}, \quad I^{**} = 2(\lambda_0 + \mu)J^{**}. \tag{33}$$

From equalities (29) and (33) it follows that the average shear strain ($\varepsilon_{r\theta}^{**} = 0$) is absent and the average invariants are proportional. The same equalities imply the relations $2P_{rr}^{**} = 2P_{\theta\theta}^{**} = I^{**}$ and $2\varepsilon_{rr}^{**} = 2\varepsilon_{\theta\theta}^{**} = J^{**}$, which, together with the relation for the invariants (33), establishes the proportionality between corresponding average components of stresses and strains

$$P_{rr}^{**} = 2(\lambda_0 + \mu)\varepsilon_{rr}^{**}, \qquad P_{\theta\theta}^{**} = 2(\lambda_0 + \mu)\varepsilon_{\theta\theta}^{**}. \tag{34}$$

Thus, for the diagonal components (34), the coefficients of proportionality are identical and equal to those between the invariants (32).

The contour stresses (30) correspond to the components ε_{rr}^1, $\varepsilon_{\theta\theta}^1$, $\varepsilon_{r\theta}^1$ and invariant J^1 of the contour strains. They are related by equalities (31) and (32) written for $r = 1$:

$$P_{rr}^1 = \lambda_0 J^1 + 2\mu\varepsilon_{rr}^1, \qquad P_{\theta\theta}^1 = \lambda_0 J^1 + 2\mu\varepsilon_{\theta\theta}^1, \qquad P_{r\theta}^1 = 2\mu\varepsilon_{r\theta}^1; \tag{35}$$

$$I^1 = 2(\lambda_0 + \mu)J^1, \qquad I^1 = P_{rr}^1 + P_{\theta\theta}^1, \qquad J^1 = \varepsilon_{rr}^1 + \varepsilon_{\theta\theta}^1. \tag{36}$$

Relations (30), (35), and (36) imply that the shear strain vanishes on the contour ($\varepsilon_{r\theta}^1 = 0$), the stress and strain invariants are proportional to each other [equality (36)], and the diagonal components of stresses and strains are proportional:

$$P_{rr}^1 = -kP_{\theta\theta}^1, \qquad ((1+k)\lambda_0 + 2\mu)\varepsilon_{rr}^1 = -((1+k)\lambda_0 + 2k\mu)\varepsilon_{\theta\theta}^1.$$

From these equalities, we obtain the following expression of the invariants in terms of the diagonal components:

$$I^1 = -\frac{1-k}{k}P_{rr}^1 = (1-k)P_{\theta\theta}^1, \qquad J^1 = -\frac{2\mu(1-k)}{(1+k)\lambda_0 + 2k\mu}\varepsilon_{rr}^1 = \frac{2\mu(1-k)}{(1+k)\lambda_0 + 2\mu}\varepsilon_{\theta\theta}^1. \tag{37}$$

Substituting (37) into (36), we obtain

$$P_{rr}^1 = \frac{4k\mu(\lambda_0 + \mu)}{(1+k)\lambda_0 + 2k\mu}\varepsilon_{rr}^1, \qquad P_{\theta\theta}^1 = \frac{4\mu(\lambda_0 + \mu)}{(1+k)\lambda_0 + 2\mu}\varepsilon_{\theta\theta}^1. \tag{38}$$

Relations (38) are similar to relations (34) and differ from the latter only by the coefficients. The coefficients in (34) are the same and depend only on the elastic constants, whereas the coefficients in (38) are different and contain the parameter k. This parameter, in view of (24), can be expressed in terms of the parameter m: $k = (\chi - m)/(\chi + m)$; therefore, the coefficients in (38) take into account not only the elastic properties but also the shape of the hole.

3. The adopted assumption of uniform stresses in the thread and the introduced characteristics of the stress field in the plate, which are independent of the point, make it possible to model the mechanical behavior of the thread in tension using one of the following assumptions.

The thread oval has small thickness and, therefore, the stress τ and the strain e in it are normal components of the stress and strain tensors at the sites that are orthogonal to the oval axis. This axis is close to the inner boundary of the oval, which is the θ axis of the elliptic coordinates in the plate. Consequently, the quantities τ and e at the above-mentioned sites correspond to the quantities $P_{\theta\theta}$ and $\varepsilon_{\theta\theta}$ on the hole contour of the plate. Therefore, it can be assumed that the measures of the stress strain state of the oval τ^* and e^* are determined by the corresponding averaged measures of the plate $P^{**}_{\theta\theta}$ and $\varepsilon^{**}_{\theta\theta}$, and the relation (34) between these quantities gives the following law of mechanical behavior of the thread (the first model):

$$\tau^* = \alpha^* e^*, \quad \tau^* = P^{**}_{\theta\theta}, \quad e^* = \varepsilon^{**}_{\theta\theta}, \quad \alpha^* = 2(\lambda_0 + \mu). \tag{39}$$

Alternatively, the characteristics of the oval τ^1 and e^1 are determined by the contour quantities $P^1_{\theta\theta}$ and $\varepsilon^1_{\theta\theta}$ in the plate, and, hence, the behavior of the thread is expressed by relation (38) (second model):

$$\tau^1 = \alpha^1 e^1, \quad \tau^1 = P^1_{\theta\theta}, \quad e^1 = \varepsilon^1_{\theta\theta}, \quad \alpha^1 = 4\mu(\lambda_0 + \mu)/((1+k)\lambda_0 + 2\mu). \tag{40}$$

One can also assume that the stress and strain of the oval τ^0 and e^0 are determined by the invariants I and J of the stress and strain fields in the plate, and the behavior of the thread is described by relation (31) (third model):

$$\tau^0 = \alpha^0 e^0, \quad \tau^0 = I, \quad e^0 = J, \quad \alpha^0 = 2(\lambda_0 + \mu). \tag{41}$$

Models (39)–(41) have a similar structure. The coefficients of proportionality in them possess the properties $\alpha^0 = \alpha^* \neq \alpha^1$; therefore, relations (39) and (41) are identical and relations (39) and (40) are alternative. By virtue of this, we shall consider only models (39) and (40). The property of these coefficients $\alpha^*/\alpha^1 = 1 + \lambda_0(1+k)/(2\mu) > 1$ shows that, in comparison with (40), the law (39) gives more rapid growth of stresses.

Each of models (39) and (40) allows one to express the strain of the thread in terms of specified quantities and quantities determined experimentally. Indeed, in these models, the stresses are known and, in accordance with (2), (9), (17), (29), and (30), they are given by the formulas

$$\tau^* = P^{**}_{\theta\theta} = 2A = (P^\infty_{xx} + P^\infty_{yy})/2 = (Fd + Gl)/(2hld),$$

$$\tau^1 = P^1_{\theta\theta} = 4A(1-k) = (P^\infty_{xx} + P^\infty_{yy})/(1-k) = (Fd + Gl)/((1-k)hld).$$

Using the relations between different elastic constants [6, 12]

$$\lambda = E\nu/((1-2\nu)(1+\nu)), \qquad \mu = E/(2(1+\nu)),$$

where E is the Young's modulus and ν is the Poisson's ratio (determined experimentally), it is possible to express the coefficients of proportionality in the models considered in the form $\alpha^* = E/(1-\nu)$ and $\alpha^1 = E/(1+k\nu)$. Finally, the alternative laws of behavior of the tread (39) and (40) become

$$e^* = \frac{1-\nu}{2E}\frac{Fd+Gl}{hld}, \qquad e^1 = \frac{1+k\nu}{E(1-k)}\frac{Fd+Gl}{hld}. \tag{42}$$

Thus, according to the mathematical models proposed, the strain of the thread is determined by the geometric parameters of the cloth, the applied loads, and the strength properties of the material. Each of formulas (42) enables one to calculate the strain of the thread from specified quantities and to plot tensile diagrams.

It has been found experimentally [4, pp. 107–110] that, for biaxial extension of smooth weaved or knitted materials, they strengthen along loop rows and columns. Experimental curves of load f_i (per unit width) versus strain ε_i were approximated by the nonlinear equations $f_i = c_i\varepsilon_i^\omega$ ($i = 1, 2$), where c_i and ω

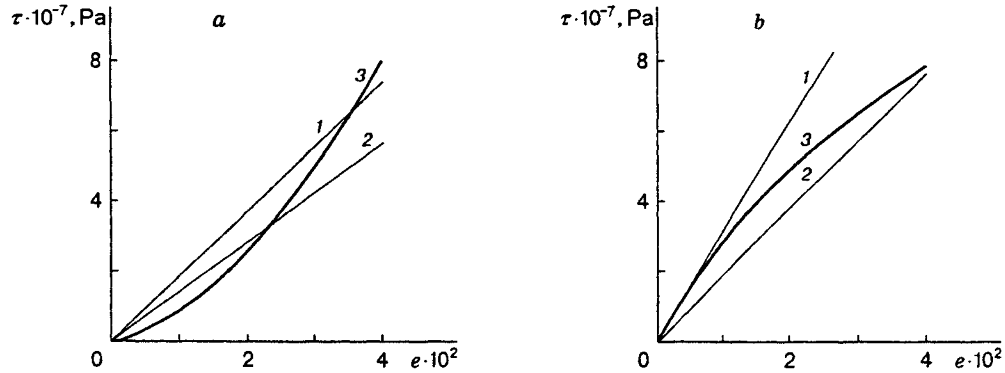

Fig. 5. Tensile curves for the yarn and knitted fabric: (a) cotton; (b) wool; curves 1 and 3 refer to the first and second models, respectively, and curves 3 to experiment.

are constants. For small strains, these relations admit linearization (based on the equalities $\varepsilon_i^{\omega} \approx e_i$) $f_i = c_i e_i$ ($i = 1, 2$) which is in good agreement with the experiment. A comparison of these functions with relations (39) and (40) shows that they are of the same type. Thus, for small strains, theoretical relations are similar to linearized experimental relations.

A comparison between theoretical and experimental results is shown in Fig. 5, where curves 3 refer to experimental results on extension of cotton and woolen yarn and theoretical straight lines 1 and 2 obtained using models 1 and 2, respectively, for extension of threads in knitted fabrics made of the same materials. For the yarn, the experimental data of [3, p. 132] were used; calculations of thread tension in the fabrics were performed using the strength characteristics given in [3, pp. 173 and 209]:

— for cotton, $E = 135 \cdot 10^7$ Pa, $\nu = 0.3$, $k = 0.1$, $\alpha^* = 193 \cdot 10^7$ Pa, and $\alpha^1 = 131 \cdot 10^7$ Pa;

— for wool, $E = 170 \cdot 10^7$ Pa, $\nu = 0.42$, $k = 0.1$, $\alpha^* = 293 \cdot 10^7$ Pa, and $\alpha^1 = 163 \cdot 10^7$ Pa.

Analysis of the results shows that the theoretical curves are sufficiently close to the experimental curves. The first model gives the upper bound of the experimental results and the second model gives the lower bound.

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